

Burnside Condition on Some Intersection Subgroups

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Abstract

In this paper, using the notions graphs, core graphs, immersions and covering maps of graphs, introduced by Stallings in 1983, we prove the Burnside condition for the intersection of subgroups of free groups with Burnside condition.

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1. Introduction and Motivation

In [3] J. Stallings studied on free groups by theory of graphs. He introduced the concept of immersions of graphs, provided an algorithmic process to study on finitely generated subgroups of free groups. Using these tools, he also gave an elegant proof for Howson's theorem "if A and B are finitely generated subgroups of a free group, then $A \cap B$ is finitely generated". Moreover, using immersions of graphs and core graphs (graphs with no trees hanging on) some mathematicians such as Everitt and Gersten studied on H. Neumann's inequality on the rank of $A \cap B$ (see [1] and [2]). Stallings also in [3], introduced another notation called "Burnside condition for a subgroup". In this paper, we focus on this notion, and using similar methods, we prove that if A and B are finitely generated subgroups of a free group F , and $A \cap B$ satisfies the Burnside condition in both A and B , then $A \cap B$ satisfies the Burnside condition in $A \vee B$, the subgroup of F generated by $A \cup B$.

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2. Preliminaries

In this section, all our notations come from [3]. A graph X consists of two sets E and V (*edges* and *vertices*), with three functions $^{-1} : E \rightarrow E$ and $s, t : E \rightarrow V$ such that $(e^{-1})^{-1} = e$, $e^{-1} \neq e$, $s(e^{-1}) = t(e)$ and $t(e^{-1}) = s(e)$. We say that the edge $e \in E$ has *initial vertex* $s(e)$ and *terminal vertex* $t(e)$. The edge e^{-1} is also called the *reverse* of e .

A *map of graphs* $f : X \rightarrow Y$ is a function which maps edges to edges and vertices to vertices. Also we have $f(e^{-1}) = f(e)^{-1}$, $f(s(e)) = s(f(e))$ and $f(t(e)) = t(f(e))$.

A *path* p in X of length $n = |p|$, with initial vertex u and terminal vertex v , is an n -tuple of edges of X of the form $p = e_1 \dots e_n$ such that for $i = 1, \dots, n-1$, we have $t(e_i) = s(e_{i+1})$ and $s(e_1) = u$ and $t(e_n) = v$. For $n = 0$, given any vertex v , there is a unique path Λ_v of length 0 whose initial and terminal vertices coincide and are equal to v . A path p is called a *circuit* if its initial and terminal vertices coincide.

If p and q are paths in X and the terminal vertex of p equals the initial vertex of q , they may be *concatenated* to form a path pq with $|pq| = |p| + |q|$, whose initial vertex is that of p and whose terminal vertex is that of q .

A *round-trip* is a path of the form ee^{-1} . A *reduced path* is a path in X containing no round-trip. An *elementary reduction* is insertion or deletion a round-trip in a path. Two paths p and q are *homotopic* (written $p \sim q$) iff there is a finite sequence of elementary reductions taking one path to the other. Homotopic paths must have the same start and terminal vertices and also, homotopy is an equivalence relation on the set of paths with same start and same terminal vertices in X . Moreover, any path in X is homotopic to a unique reduced path in X .

Let v be a fix vertex in X , $\pi_1(X, v)$ is defined to be the set of all homotopy classes of closed paths with initial and terminal vertex v . Then $\pi_1(X, v)$ together with the product $[p][q] := [pq]$ forms a group with identity $[\Lambda_v]$ and inverse element $[p]^{-1} = [p^{-1}]$.

For a fix vertex v in X , the *star* of v in X is defined as follows:

$$St(v, X) = \{e \in E : s(e) = v\}.$$

A map $f : X \rightarrow Y$ yields, for each vertex $v \in X$, a function $f_v : St(v, X) \rightarrow St(f(v), Y)$. If for each vertex $v \in X$, f_v is injective, we call f an *immersion* and if for each vertex $v \in X$, f_v is surjective, we call f a *locally surjective*. If each f_v is bijective, we call f a *covering map*.

The theory of coverings of graphs is almost completely analogous to the topological theory of coverings. Immersions have some of the properties of coverings, one of them we need it more, is the following one:

”For a given finite set of elements $\{\alpha_1, \dots, \alpha_n\} \subseteq \pi_1(X, u)$, there is a connected graph Y and an immersion $f : Y \rightarrow X$ such that $f_*(\pi_1(Y)) = S$, in which S is the subgroup of $\pi_1(X, u)$ generated by $\{\alpha_1, \dots, \alpha_n\}$ ”.

If G is a group, a G -graph X is a graph with an action of G on the left on X by maps of graphs, such that for all $g \in G$ and every edge e , $ge \neq e^{-1}$. In this case, the quotient graph X/G , and the natural quotient map of graphs $q : X \rightarrow X/G$ are defined. It is easy to see that, in general $q : X \rightarrow X/G$ is locally surjective.

We call G *acts freely* on X , whenever v is a vertex of X , $g \in G$, and $gv = v$, then $g = 1$, the identity element of G . In this case, $q : X \rightarrow X/G$ is an immersion, and hence is a covering map.

A *translation* of a map of graphs $f : X \rightarrow Y$ is a map $g : X \rightarrow X$ which is an isomorphism of graphs and for which $fg = f$. The set of all translations of f forms a group $G(f)$ which acts on X . If f is an immersion, and X is connected, then $G(f)$ acts freely on X .

For a connected graph X , the *universal cover* $f : \tilde{X} \rightarrow X$ is a covering map with \tilde{X} connected and $\pi_1(\tilde{X})$ is trivial.

Lemma 2.1. [3] *If $f : \tilde{X} \rightarrow X$ is a universal covering map, then $G(f) \cong \pi_1(X)$ which acts freely, by covering translations, on \tilde{X} , and also f is isomorphic to the quotient map $q : \tilde{X} \rightarrow \tilde{X}/G$, i.e., there exists an isomorphism $\varphi : \tilde{X}/G \rightarrow X$ such that $\varphi q = f$.*

Theorem 2.2. [3] *Let*

$$\begin{array}{ccc} Z_3 & \xrightarrow{g_1} & Z_1 \\ \downarrow g_2 & & \downarrow f_1 \\ Z_2 & \xrightarrow{f_2} & X. \end{array}$$

be a pullback diagram of graphs, where f_1 and f_2 are immersions. Let v_1 and v_2 be vertices in Z_1 and Z_2 that $f_1(v_1) = f_2(v_2) = w$. Let v_3 be corresponding vertex in Z_3 . Define $f_3 = f_1g_1 = f_2g_2 : Z_3 \rightarrow X$, and $S_i = f_{i}(\pi_1(Z_i, v_i))$, for $i = 1, 2, 3$. Then $S_3 = S_1 \cap S_2$.*

3. Main results

In this section, we deduce our main result. Before it, we recall some notes from [3] which are essential in our proof. First, we note to the *core graphs* whose roles are more important.

A *cyclically reduced circuit* in a graph X is a circuit $p = e_1 \dots e_n$, which is reduced as a path and for which $e_1 \neq e_n^{-1}$. A graph X is said to be a *core-graph* if X

is connected, has at least one edge and each of its edges belongs to at least one cyclically reduced circuit.

In a connected graph X with non-trivial fundamental group, an *essential edge* is an edge belonging to some cyclically reduced circuit. The *core* of X consists of all essential edges of X and all initial vertices of essential edges.

If X is a connected graph with non-trivial fundamental group, then the core X' of X is a core-graph. If v is a vertex of X' , then the homomorphism induced by inclusion, $\pi_1(X', v) \rightarrow \pi_1(X, v)$ is an isomorphism.

Another notion, we are dealing with, is the *Burnside condition* for subgroups. A subgroup S of a group G satisfies the *Burnside condition* if for every $g \in G$, there exists some positive integer n such that $g^n \in S$.

Lemma 3.1. [3] (a) *Let $f : X \rightarrow Y$ be a finite-sheeted covering of connected graphs, and v a vertex of X . Then $f_*(\pi_1(X, v)) \subseteq \pi_1(Y, f(v))$ satisfies the Burnside condition.*

(b) *Let $f : X \rightarrow Y$ be an immersion of connected graphs, Y be a core-graph, v a vertex of X , and $f_*(\pi_1(X, v)) \subseteq \pi_1(Y, f(v))$ satisfy the Burnside condition. Then f is a covering map.*

Finally, using all the above notes, we establish our main result in the follow.

Theorem 3.2. *Let S_1 and S_2 be finitely generated subgroups of a free group F . If $S_1 \cap S_2$ satisfies the Burnside condition both in S_1 and S_2 , then $S_1 \cap S_2$ also satisfies the Burnside condition in $S_1 \vee S_2$, the subgroup generated by $S_1 \cup S_2$.*

Proof. Similar to the argument of 7.8 in [3], we start with the following pullback diagram of immersions

$$\begin{array}{ccc} Z_3 & \xrightarrow{g_1} & Z_1 \\ \downarrow g_2 & & \downarrow f_1 \\ Z_2 & \xrightarrow{f_2} & X. \end{array}$$

where f_1 and f_2 are immersions, Z_3 is a subgraph of the pullback; Z_1 , Z_2 and Z_3 are core-graphs; $f_3 = f_1g_1 = f_2g_2$; v_3 is a vertex of Z_3 , v_1 and v_2 are the images of v_3 in Z_1 and Z_2 ; $w = f_i(v_i)$ ($i = 1, 2, 3$), $F = \pi_1(X, w)$, $S_i = f_{i*}(\pi_1(Z_i, v_i))$ ($i = 1, 2, 3$). Using Theorem 2.2, $S_3 = S_1 \cap S_2$.

By Lemma 3.1 (b), since S_3 satisfies the Burnside condition both in S_1 and S_2 , it follows that the immersions g_1 and g_2 are coverings.

Let $r : \tilde{Z}_3 \rightarrow Z_3$ be a universal covering, and consider $\tilde{g}_1 = g_1r$ and $\tilde{g}_2 = g_2r$, which are consequently universal coverings. Then by Lemma 2.1, $G(\tilde{g}_1) = S_1$, $G(\tilde{g}_2) = S_2$

and the universal coverings $\tilde{g}_1 : \tilde{Z}_3 \rightarrow Z_1$ and $\tilde{g}_2 : \tilde{Z}_3 \rightarrow Z_2$ are isomorphic to quotient maps $q_1 : \tilde{Z}_3 \rightarrow \tilde{Z}_3/G(\tilde{g}_1)$ and $q_2 : \tilde{Z}_3 \rightarrow \tilde{Z}_3/G(\tilde{g}_2)$, respectively, i.e., there are isomorphisms $\varphi_1 : \tilde{Z}_3/G(\tilde{g}_1) \rightarrow Z_1$ and $\varphi_2 : \tilde{Z}_3/G(\tilde{g}_2) \rightarrow Z_2$ such that $\varphi_1 q_1 = \tilde{g}_1$ and $\varphi_2 q_2 = \tilde{g}_2$. By definition, for any covering transformation $\sigma \in G(\tilde{g}_1) = S_1$, $\tilde{g}_1 \sigma = \tilde{g}_1$, and so $g_1 r \sigma = g_1 r$. Hence, $f_3 r \sigma = f_3 r = h$. So σ is a transformation of the immersion h . Similarly, any covering transformation of \tilde{g}_2 is a transformation of h . Now, suppose K is the group of transformations of h generated by $G(\tilde{g}_1) \cup G(\tilde{g}_2)$, then $K \cong S_1 \vee S_2$ and we have the following pushout diagram.

$$\begin{array}{ccc} \tilde{Z}_3 & \xrightarrow{q_1} & \tilde{Z}_3/G(\tilde{g}_1) \\ q_2 \downarrow & \searrow t_3 & \downarrow t_1 \\ \tilde{Z}_3/G(\tilde{g}_2) & \xrightarrow{t_2} & \tilde{Z}_3/K \end{array}$$

Therefore, because of the universal property of pushout, for graph X and maps $f_1 \varphi_1$ and $f_2 \varphi_2$ there exists a unique map $s : \tilde{Z}_3/K \rightarrow X$ such that $st_1 = f_1 \varphi_1$ and $st_2 = f_2 \varphi_2$.

Since h is an immersion, then $G(h)$ and hence K acts freely on \tilde{Z}_3 , and so t_3 is a covering. It follows that t_1 is a covering too. $\tilde{Z}_3/G(\tilde{g}_1)$ is a finite graph and so t_1 is a finite-sheeted covering. Thus by Lemma 3.1 (a), $t_{1*}(\pi_1(\tilde{Z}_3/G(\tilde{g}_1)))$ satisfies the Burnside condition in $\pi_1(\tilde{Z}_3/K)$; so $s_*(t_{1*}(\pi_1(\tilde{Z}_3/G(\tilde{g}_1)))) = f_{1*}(\varphi_{1*}(\pi_1(\tilde{Z}_3/G(\tilde{g}_1)))) = f_{1*}(\pi_1(Z_1)) = S_1$ satisfies the Burnside condition in $s_*(\pi_1(\tilde{Z}_3/K))$. As we can see, $s_*(\pi_1(\tilde{Z}_3/K))$ contains S_1 and similarly S_2 and hence it contains $S_1 \vee S_2$. Therefore, since $S_1 \subseteq S_1 \vee S_2 \subseteq s_*(\pi_1(\tilde{Z}_3/K))$ and S_1 satisfies the Burnside condition in $s_*(\pi_1(\tilde{Z}_3/K))$, then S_1 also satisfies the Burnside condition in $S_1 \vee S_2$. Now, by the fact that $S_1 \cap S_2$ satisfies the Burnside condition in S_1 , the result holds. \square

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